

REGULAR COVERS OF OPEN RELATIVELY COMPACT SUBANALYTIC SETS

ADAM PARUSIŃSKI

Let M be a real analytic manifold of dimension n . In this paper we study the algebra $\mathcal{S}(M)$ of relatively compact open subanalytic subsets of M . As we show this algebra is generated by sets with Lipschitz regular boundaries. More precisely, we call a relatively compact open subanalytic subset $U \subset M$ an *open subanalytic Lipschitz ball* if its closure is subanalytically bi-Lipschitz homeomorphic to the unit ball of \mathbb{R}^n . Here we assume that M is equipped with a Riemannian metric. Any two such metrics are equivalent on relatively compact sets and hence the above definition is independent of the choice of a metric.

Theorem 0.1. *The algebra $\mathcal{S}(M)$ is generated by open subanalytic Lipschitz balls.*

That is to say if U is a relatively compact open subanalytic subset of M then the characteristic function 1_U is a linear combination of functions of the form $1_{W_1}, \dots, 1_{W_m}$, where the W_j are open subanalytic Lipschitz balls. Note that, in general, U cannot be covered by subanalytic Lipschitz balls, as it is easy to see for $\{(x, y) \in \mathbb{R}^2; y^2 < x^3, x < 1\}$, $M = \mathbb{R}^2$, due to the presence of cusps. Nevertheless we show the existence of a "regular" cover in the sense that we control the distance to the boundary.

Theorem 0.2. *Let $U \in \mathcal{S}(M)$. Then there exist a finite cover $U = \bigcup_i U_i$ by open subanalytic sets such that :*

- (1) *every U_i is subanalytically homeomorphic to an open n -dimensional ball;*
- (2) *there is $C > 0$ such that for every $x \in U$, $\text{dist}(x, M \setminus U) \leq C \max_i \text{dist}(x, M \setminus U_i)$*

The proofs of Theorems 0.1 and 0.2 are based on the regular projection theorem, cf. [6], [7], [8], the classical cylindrical decomposition, and the L-regular decomposition of subanalytic sets, cf. [4], [8], [9]. L-regular sets are natural multidimensional generalization of classical cusps. We recall them briefly in Subsection 1.6. We show also the following strengthening of Theorem 0.2.

Theorem 0.3. *In Theorem 0.2 we may require additionally that all U_i are open L-regular cells (i.e. interiors of L-regular sets).*

For an open $U \subset M$ we denote $\partial U = \overline{U} \setminus U$.

1. PROOFS

1.1. Reduction to the case $M = \mathbb{R}^n$. Let $U \in \mathcal{S}(M)$. Choose a finite covering $\overline{U} \subset \bigcup_i V_i$ by open relatively compact sets such that for each V_i there is an open neighborhood of $\overline{V_i}$ analytically diffeomorphic to \mathbb{R}^n . Then there are finitely many open subanalytic U_{ij} such that $U_{ij} \subset V_i$ and 1_U is a combination of $1_{U_{ij}}$. Thus it suffices to show Theorem 0.1 for relatively compact open subanalytic subsets of \mathbb{R}^n .

Similarly, it suffices to show Theorems 0.2 and 0.3 for $M = \mathbb{R}^n$. Indeed, it follow from the observation that the function

$$x \rightarrow \max_i \text{dist}(x, M \setminus V_i)$$

is continuous and nowhere zero on $\bigcup_i V_i$ and hence bounded from below by a nonzero constant $c > 0$ on \overline{U} . Then

$$\text{dist}(x, M \setminus U) \leq C_1 \leq c^{-1} C_1 \max_i \text{dist}(x, M \setminus V_i)$$

where C_1 is the diameter of \overline{U} and hence, if $c^{-1} C_1 \geq 1$,

$$\text{dist}(x, M \setminus U) \leq c^{-1} C_1 \max_i (\min\{\text{dist}(x, M \setminus U), \text{dist}(x, M \setminus V_i)\}).$$

Now if for each $U \cap V_i$ we choose a covering U_{ij} satisfying the statement of Theorem 0.2 or 0.3 then for $x \in U$

$$\begin{aligned} \text{dist}(x, M \setminus U) &\leq c^{-1} C_1 \max_i (\min\{\text{dist}(x, M \setminus U), \text{dist}(x, M \setminus V_i)\}) \\ &\leq c^{-1} C_1 \max_i \text{dist}(x, M \setminus U \cap V_i) \leq C c^{-1} C_1 \max_{ij} \text{dist}(x, M \setminus U_{ij}) \end{aligned}$$

Thus the cover U_{ij} satisfies the claim of Theorem 0.2, resp. Theorem 0.3.

1.2. Regular projections. We recall after [7], [8] the subanalytic version of the regular projection theorem of T. Mostowski introduced originally in [6] for complex analytic sets germs.

Let $X \subset \mathbb{R}^n$ be subanalytic. For $\xi \in \mathbb{R}^{n-1}$ we denote by $\pi_\xi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ the linear projection parallel to $(\xi, 1) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Fix constants $C, \varepsilon > 0$. We say that $\pi = \pi_\xi$ is (C, ε) -regular at $x_0 \in \mathbb{R}^n$ (with respect to X) if

- (a) $\pi|_X$ is finite;
- (b) the intersection of X with the open cone

$$(1.1) \quad \mathcal{C}_\varepsilon(x_0, \xi) = \{x_0 + \lambda(\eta, 1); |\eta - \xi| < \varepsilon, \lambda \in \mathbb{R} \setminus 0\}$$

is empty or a finite disjoint union of sets of the form

$$\{x_0 + \lambda_i(\eta)(\eta, 1); |\eta - \xi| < \varepsilon\},$$

where λ_i are real analytic nowhere vanishing functions defined on $|\eta - \xi| < \varepsilon$.

- (c) the functions λ_i from (b) satisfy for all $|\eta - \xi| < \varepsilon$

$$\|\text{grad } \lambda_i(\eta)\| \leq C |\lambda_i(\eta)|,$$

We say that $\mathcal{P} \subset \mathbb{R}^{n-1}$ defines a set of regular projections for X if there exists $C, \varepsilon > 0$ such that for every $x_0 \in \mathbb{R}^n$ there is $\xi \in \mathcal{P}$ such that π_ξ is (C, ε) -regular at x_0 .

Theorem 1.1 ([7], [8]). *Let X be a compact subanalytic subset of \mathbb{R}^n such that $\dim X < n$. Then the generic set of $n + 1$ vectors ξ_1, \dots, ξ_{n+1} , $\xi_i \in \mathbb{R}^{n-1}$, defines a set of regular projections for X .*

(Here by generic we mean in the complement of a subanalytic nowhere dense subset of $(\mathbb{R}^{n-1})^{n+1}$.)

1.3. Cylindrical decomposition. We recall the first step of a basic construction, the cylindrical algebraic decomposition, for details see for instance [2], [3].

Set $X = \overline{U} \setminus U$. Then X is a compact subanalytic subset of \mathbb{R}^n of dimension $n - 1$. We denote by $Z \subset X$ the set of singular points of X that is the complement in X of the set

$$\text{Reg}(X) := \{x \in X; (X, x) \text{ is the germ of a real analytic submanifold of dimension } n - 1\}.$$

Then Z is closed in X , subanalytic and $\dim Z \leq n - 2$.

Assume that the standard projection $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ restricted to X is finite. Denote by $\Delta_\pi \subset \mathbb{R}^{n-1}$ the union of $\pi(Z)$ and the set of critical values of $\pi|_{\text{Reg}(X)}$. Then Δ_π , called *the discriminant set of π* , is compact and subanalytic. It is clear that $\overline{\pi(U)} = \pi(U) \cup \Delta_\pi$.

Proposition 1.2. *Let $U' \subset \pi(U) \setminus \Delta_\pi$ be open and connected. Then there are finitely many bounded real analytic functions $\varphi_1 < \varphi_2 < \dots < \varphi_k$ defined on U' , such that $X \cap \pi^{-1}(U')$ is the union of graphs of φ_i 's. In particular, $U \cap \pi^{-1}(U')$ is the union of the sets*

$$\{(x', x_n) \in \mathbb{R}^n; x' \in U', \varphi_i(x') < x_n < \varphi_{i+1}(x'),\}$$

and moreover, if U' is subanalytically homeomorphic to an open $(n - 1)$ -dimensional ball, then each of these sets is subanalytically homeomorphic to an open n -dimensional ball.

1.4. The case of a regular projection. Fix $x_0 \in U$ and suppose that $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ is (C, ε) -regular at $x_0 \in \mathbb{R}^n$ with respect to X . Then the cone (1.1) contains no point of Z . By [8] Lemma 5.2, this cone contains no critical point of $\pi|_{\text{Reg}(X)}$, provided ε is chosen sufficiently small (for fixed C). In particular, $x'_0 = \pi(x_0) \notin \Delta_\pi$.

In what follows we fix $C, \varepsilon > 0$ and suppose ε small. We denote the cone (1.1) by \mathcal{C} for short. Then for \tilde{C} sufficiently large, that depends only on C and ε , we have

$$(1.2) \quad \text{dist}(x_0, X \setminus \mathcal{C}) \leq \tilde{C} \text{dist}(x'_0, \pi(X \setminus \mathcal{C})) \leq \tilde{C} \text{dist}(x'_0, \Delta_\pi).$$

The first inequality is obvious, the second follow from the fact that the singular part of X and the critical points of $\pi|_{\text{Reg}(X)}$ are both outside the cone.

1.5. Proof of Theorem 0.2. Induction on n . Set $X = \overline{U} \setminus U$ and let $\pi_{\xi_1}, \dots, \pi_{\xi_{n+1}}$ be a set of (C, ε) -regular projections with respect to X . To each of these projections we apply the cylindrical decomposition. More precisely, let us fix one of these projections that for simplicity we suppose standard and denote it by π . Then we apply the inductive assumption to $\pi(U) \setminus \Delta_\pi$. Thus let $\pi(U) \setminus \Delta_\pi = \bigcup U'_i$ be a finite cover satisfying the statement of Theorem 0.2. Applying to each U'_i Proposition 1.2 we obtain a family of cylinders that cover $U \setminus \pi^{-1}(\Delta_\pi)$. In particular they cover the set of those points of U at which π is (C, ε) -regular.

Lemma 1.3. *Suppose π is (C, ε) -regular at $x_0 \in U$. Let U' be an open subanalytic subset of $\pi(U) \setminus \Delta_\pi$ such that $x'_0 = \pi(x_0) \in U'$ and*

$$(1.3) \quad \text{dist}(x'_0, \Delta_\pi) \leq \tilde{C} \text{dist}(x'_0, \partial U'),$$

with $\tilde{C} \geq 1$ for which (1.2) holds. Then

$$(1.4) \quad \text{dist}(x_0, X) \leq (\tilde{C})^2 \text{dist}(x_0, \partial U_1),$$

where U_1 is the member of cylindrical decomposition of $U \cap \pi^{-1}(U')$ containing x_0 .

Proof. We decompose ∂U_1 into two parts. The first one is vertical, i.e. contained in $\pi^{-1}(\partial U')$ and the second part is contained in X . The distance to the first one from x_0 equals to the horizontal distance, that is $\text{dist}(x'_0, \partial U')$. Thus we have

$$(1.5) \quad \text{dist}(x_0, \partial U_1) = \min\{\text{dist}(x_0, X), \text{dist}(x'_0, \partial U')\}.$$

If $\text{dist}(x_0, \partial U_1) = \text{dist}(x_0, X)$ then (1.4) holds with $\tilde{C} = 1$, otherwise $\text{dist}(x_0, \partial U_1) = \text{dist}(x'_0, \partial U') \leq \text{dist}(x_0, X)$ and then by (1.3) and (1.2)

$$(1.6) \quad \text{dist}(x_0, X \setminus \mathcal{C}) \leq \tilde{C} \text{dist}(x'_0, \Delta_\pi) \leq (\tilde{C})^2 \text{dist}(x'_0, \partial U') \leq (\tilde{C})^2 \text{dist}(x_0, X).$$

□

Thus to complete the proof of Theorem 0.2 it suffices to show that the assumptions of Lemma 1.3 are satisfied. This follows from the inclusion $\partial\pi(U) \subset \Delta_\pi$ that gives $\text{dist}(x'_0, \Delta_\pi) \leq \text{dist}(x'_0, \partial\pi(U))$, and from $\text{dist}(x'_0, \partial\pi(U)) \leq \tilde{C} \text{dist}(x'_0, \partial U')$ that holds by the inductive assumption. This ends the proof of Theorem 0.2.

1.6. L-regular sets. Let $Y \subset \mathbb{R}^n$ be subanalytic, $\dim Y = n$. Then Y is called *L-regular* (with respect to given system of coordinates) if

- (1) if $n = 1$ then Y is a non-empty closed bounded interval;
- (2) if $n > 1$ then Y is of the form

$$(1.7) \quad Y = \{(x', x_n) \in \mathbb{R}^n; f(x') \leq x_n \leq g(x'), x' \in Y'\},$$

where $Y' \subset \mathbb{R}^{n-1}$ is L-regular, f and g are continuous subanalytic functions defined in Y' . It is also assumed that on the interior of Y , f and g are analytic, satisfy $f < g$, and have bounded first order partial derivatives.

If $\dim Y = k < n$ then we say that Y is *L-regular* (with respect to given system of coordinates) if

$$(1.8) \quad Y = \{(y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}; z = h(y), y \in Y'\},$$

where $Y' \subset \mathbb{R}^k$ is L-regular, $\dim Y' = k$, h is a continuous subanalytic map defined on Y' , such that h is real analytic on the interior of Y , and has the first order partial derivatives bounded.

We say that Y is *L-regular* if it is L-regular with respect to a linear (or equivalently orthogonal) system of coordinates on \mathbb{R}^n .

We say that $A \subset \mathbb{R}^n$ is an *L-regular cell* if A is the relative interior of an L-regular set. That is, it is the interior of an L-regular set if $\dim A = n$, and it is the graph of h restricted to $\text{Int}(Y')$ for an L-regular set of the form (1.8). By convention, every point is a zero-dimensional L-regular cell.

By [4], see also Lemma 2.2 of [8] and Lemma 1.1 of [5], L-regular sets and L-regular cells satisfy the following property, called in [4] quasi-convexity. We say that $Z \subset \mathbb{R}^n$ is *quasi-convex* if there is a constant $C > 0$ such that every two points x, y of Z can be connected in Z by a continuous subanalytic arc of length bounded by $C\|x - y\|$. It can be shown that for an L-regular set or cell Y the constant C depends only on n and the bounds on first order partial derivatives of functions describing Y in the above definition. By Lemma 2.2 of [8], an L-regular cell is homeomorphic to the (open) unit ball.

Let Y be a subanalytic subset of a real analytic manifold M . We say that Y is L -regular if there exists its neighborhood V in M and an analytic diffeomorphism $\varphi : V \rightarrow \mathbb{R}^n$ such that $\varphi(Y)$ is L -regular. Similarly we define an L -regular cell in M .

1.7. Proof of Theorem 0.3. Fix a constant C_1 sufficiently large and a projection $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ that is assumed, for simplicity, to be the standard one. We suppose that π restricted to $X = \partial U$ is finite. We say that $x' \in \pi(U) \setminus \Delta_\pi$ is C_1 -regularly covered if there is a neighborhood \tilde{U}' of x' in $\pi(U) \setminus \Delta_\pi$ such that $X \cap \pi^{-1}(\tilde{U}')$ is the union of graphs of analytic functions with all first order partial derivatives bounded (in the absolute value) by C_1 . Denote by $U'(C_1)$ the set of all $x' \in \pi(U) \setminus \Delta_\pi$ that are C_1 regularly covered. Then $U'(C_1)$ is open (if we use strict inequalities while defining it) and subanalytic. By Lemma 5.2 of [8], if π is a (C, ε) -regular projection at x_0 then x'_0 is C_1 -regularly covered, for C_1 sufficiently big $C_1 \geq C_1(C, \varepsilon)$. Moreover we have the following result.

Lemma 1.4. *Given positive constants C, ε . Suppose that the constants \tilde{C} and C_1 are chosen sufficiently big, $C_1 \geq C_1(C, \varepsilon)$, $\tilde{C} \geq \tilde{C}(C, \varepsilon)$. Let π be (C, ε) -regular at $x_0 \notin X$ and let*

$$V' = \{x' \in \mathbb{R}^{n-1}; \text{dist}(x', x'_0) < (\tilde{C})^{-1} \text{dist}(x_0, X \cap \mathcal{C})\}.$$

Then $\pi^{-1}(V') \cap X \cap \mathcal{C}$ is the union of graphs of φ_i with all first order partial derivatives bounded (in the absolute value) by C_1 . Moreover, then either $\pi^{-1}(V') \cap (X \setminus \mathcal{C}) = \emptyset$ or

$$\text{dist}(x'_0, \Delta_\pi) = \text{dist}(x'_0, \pi(X \setminus \mathcal{C})) \leq \text{dist}(x'_0, \partial U'(C_1)).$$

Proof. We only prove the second part of the statement since the first part follows from Lemma 5.2 of [8]. If $\pi^{-1}(V') \cap X \setminus \mathcal{C} \neq \emptyset$ then any point of $\pi(X \setminus \mathcal{C})$ realizing $\text{dist}(x'_0, \pi(X \setminus \mathcal{C}))$ must be in the discriminant set Δ_π . \square

We now apply to $U'(C_1)$ the inductive hypothesis and thus assume that $U'(C_1) = \bigcup U'_i$ is a finite regular cover by open L -regular cells. Fix one of them U' and let U_1 be a member of the cylindrical decomposition of $U \cap \pi^{-1}(U')$. Then U_1 is an L -regular cell. Let $x_0 \in U_1$. We apply to x_0 Lemma 1.4.

If $\pi^{-1}(V') \cap (X \setminus \mathcal{C}) = \emptyset$ then

$$\text{dist}(x_0, X) \leq \text{dist}(x_0, X \cap \mathcal{C}) \leq \tilde{C} \text{dist}(x'_0, \partial U'(C_1)) \leq \tilde{C}^2 \text{dist}(x'_0, \partial U'),$$

where the second inequality follows from the first part of Lemma 1.4 and the last inequality by the induction hypothesis. Then $\text{dist}(x_0, X) \leq \tilde{C}^2 \text{dist}(x_0, \partial U_1)$ follows from (1.5).

Otherwise, $\text{dist}(x'_0, \Delta_\pi) \leq \text{dist}(x'_0, \partial U'(C_1)) \leq \tilde{C} \text{dist}(x'_0, \partial U')$ and the claim follows from Lemma 1.3. This ends the proof.

1.8. Proof of Theorem 0.1. The proof is based on the following result.

Theorem 1.5. [Theorem A of [4]] *Let $Z_i \subset \mathbb{R}^n$ be a finite family of subanalytic sets. Then there is a finite disjoint collection $\{A_j\}$ of L -regular cells such that each Z_i is the disjoint union of some of A_j .*

Similar results in the (more general) o-minimal set-up are proven in [5] and [9].

Let U be a relatively compact open subanalytic subset of \mathbb{R}^n . By Theorem 1.5, U is a disjoint union of L -regular cells and hence it suffices to show the statement of Theorem 0.1

for a relatively compact, not necessarily open, L-regular cell. We consider first the case of an open L-regular cell. Thus suppose that

$$(1.9) \quad U = \{(x', x_n) \in \mathbb{R}^n; f(x') < x_n < g(x'), x' \in U'\},$$

where U' is a relatively compact L-regular cell, f and g are subanalytic and analytic functions on U' with the first order partial derivatives bounded. Then, by the quasi-convexity of U' , f and g are Lipschitz. Thanks to the classical result of Banach, cf. [1] (7.5) p. 122, we may suppose that f and g are restrictions to U' of Lipschitz subanalytic functions, denoted also by f and g , defined everywhere on \mathbb{R}^{n-1} and satisfying $f \leq g$. Indeed, Banach gives the following formula for such an extension of a Lipschitz function f defined on a subset of a metric space

$$\tilde{f}(p) = \sup_{q \in B} f(q) - L\|p - q\|,$$

where L is the Lipschitz constant of f . Then \tilde{f} is Lipschitz with the same constant as f and subanalytic if so was f . By the inductive assumption on dimension we may assume that U is given by (1.9) with U' an L-regular cell. Denote U by $U_{f,g}$ to stress its dependence on f and g (with U' fixed). Then

$$1_{U_{f,g}} = 1_{U_{f-1,g}} + 1_{U_{f,g+1}} - 1_{U_{f-1,g+1}}$$

and $U_{f-1,g}$, $U_{f,g+1}$ and $U_{f-1,g+1}$ are open subanalytic Lipschitz balls.

Suppose now that

$$(1.10) \quad U = \{(y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}; z = h(y), y \in U'\},$$

where U' is a relatively compact open L-regular cell of \mathbb{R}^k , h is a subanalytic and analytic map defined on U' with the first order partial derivatives bounded. Hence h is Lipschitz. We may again assume that h is the restriction of a Lipschitz subanalytic map $h : \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$ and then, by the inductive hypothesis, that U' is a subanalytic Lipschitz ball. Let

$$U_\emptyset = \{(y, z) \in U' \times \mathbb{R}^{n-k}; h_i(y) - 1 < z_i < h_i(y) + 1, i = 1, \dots, n-k\}$$

For $I \subset \{1, \dots, n-k\}$ we denote

$$U_I = \{(y, z) \in U_\emptyset; z_i \neq h_i(y) \text{ for } i \in I\}.$$

Note that each U_I is the disjoint union of $2^{|I|}$ of open subanalytic Lipschitz balls and that

$$1_U = \sum_{I \subset \{1, \dots, n-k\}} (-1)^{|I|} 1_{U_I}.$$

This ends the proof.

REFERENCES

- [1] S. Banach, *Wstęp do teorii funkcji rzeczywistych*, Monografie Matematyczne, Warszawa-Wrocław, 1951, (in Polish)
- [2] M. Coste, *An introduction to o-minimal geometry*, Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Matematica, Istituti Editoriali e Poligrafici Internazionali, Pisa (2000), <http://perso.univ-rennes1.fr/michel.coste/polysens/OMIN.pdf>
- [3] L. van den Dries, *Tame Topology and O-minimal Structures*, London Math. Soc. Lecture Note **248**. Cambridge Univ. Press 1998

- [4] K. Kurdyka, *On a Subanalytic Stratification Satisfying a Whitney Property with Exponent 1*, in Real Algebraic Geometry, Proceedings, Rennes 1991, Lecture Notes in Math. **1524**, eds M. Coste et al., (1992), 316–323
- [5] K. Kurdyka, A. Parusinski, *Quasi-convex decomposition in o-minimal structures. Application to the gradient conjecture*. Singularity theory and its applications, 137177, Adv. Stud. Pure Math., 43, Math. Soc. Japan, Tokyo, 2006.
- [6] T. Mostowski, *Lipschitz Equisingularity*, Dissertationes Math. **243** (1985).
- [7] A. Parusiński, *Regular projection for sub-analytic sets*, C. R. Acad. Sci. Paris, Série I **307**, (1988), 343–347.
- [8] A. Parusiński, *Lipschitz stratification of subanalytic sets*, Annales Sci. Éc. Norm. Sup. **27**, 6 (1994), 661–696
- [9] W. Pawłucki, *Lipschitz cell decomposition in o-minimal structures*. I. Illinois J. Math. 52 (2008), no. 3, 10451063

LABORATOIRE J.A. DIEUDONNÉ UMR CNRS 7351, UNIVERSITÉ DE NICE - SOPHIA ANTIPOLIS, PARC VALROSE, 06108 NICE CEDEX 02, FRANCE

E-mail address: adam.parusinski@unice.fr